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Relaxed Controls for Functional Equations*

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1. INTRODUCTION

We wish to study a class of variational problems defined by functional equations and, in particular, by nonlinear integral equations. Special problems of this kind, involving one-dimensional "hereditary" and delay-differential equations were investigated, among others, by A. Friedman [*1*], M. N. Oğuztöreli [*2*], and A. Halanay [*3*] (see also [*2*] and [*3*] for other references to work on such one-dimensional problems). Control problems defined by multi-dimensional integral equations were discussed in a heuristic manner by A. G. Butkovskii [*4*]. The "usual" control problems, defined by ordinary differential equations, also represent a special case. The methods that we employ are closely related to those previously developed in [*5*] and [*6*].

As a convenient framework for our study we consider the following problem: Let $\mathscr{Y}$ and Q be given spaces, $\mathscr{Y}$ Hausdorff, $\mathscr{U}$ a subset of Q, E_m the euclidean m-space, B_1 a convex closed subset of E_m, and $F : \mathscr{Y} \times Q \to \mathscr{Y}$ and $c = (c^1,..., c^m) : \mathscr{Y} \times Q \to E_m$ given functions. The "original problem" consists in determining an "original minimizing point", that is, a point $(\bar{y}, \bar{u}) \in \mathscr{Y} \times \mathscr{U}$ that minimizes $c^1(y, u)$ on the set $\{(y, u) \in \mathscr{Y} \times \mathscr{U} \mid y = F(y, u),\ c(y, u) \in B_1\}$; the "relaxed problem" consists in determining a "relaxed minimizing point" $(\bar{y}, \bar{q})$ and an "approximate minimizing solution" $\{(y_i, u_i)\}_{i=1}^{\infty}$, that is, a point $(\bar{y}, \bar{q}) \in \mathscr{Y} \times Q$ that minimizes $c^1(y, q)$ on the set

$$\{(y, q) \in \mathscr{Y} \times Q \mid y = F(y, q),\ c(y, q) \in B_1\},$$

and a sequence $\{(y_i, u_i)\}_{i=1}^{\infty}$ in $\mathscr{Y} \times \mathscr{U}$ such that $y_i = F(y_i, u_i)$ and $\lim_{i\to\infty} c(y_i, u_i) = c(\bar{y}, \bar{q})$.

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This formulation is motivated by a typical model of a control problem : The parameter u describes the control functions and parameters (that can be chosen from some "admissible" set $\mathscr{U}$), the point y describes a motion of the system consistent with the chosen controls and subject to the "equation of motion"

$$y = F(y, u), \tag{1.1}$$

the relation

$$c(y, u) \in B_1 \tag{1.2}$$

describes the restrictions imposed on the system, and c^1 is the cost functional.

In general, as in the special case of variational problems defined by ordinary differential equations, the original problem, with controls in $\mathscr{U}$, does not admit a minimizing solution even if the functions F and c are "nice". We therefore embed $\mathscr{U}$ in a set Q of "relaxed controls" and define an appropriate topology on Q in which $\mathscr{U}$ is a dense subset of sequentially compact Q and F and c are continuous when restricted to the set $\{(y, q) \in \mathscr{Y} \times Q \mid y = F(y, q)\}$. This insures, subject to certain mild assumptions about F and c, the existence of a relaxed minimizing point $(\bar{y}, \bar{q})$ and of an approximate minimizing solution. The desired "relaxed" behavior of the system can be simulated by using an element of an approximate minimizing solution.

In studying necessary conditions for minimum we require additional assumptions related to the nature of $\mathscr{Y}$ as a Banach space, the convexity of Q, the existence of (Frechet) derivatives F_y and c_y, and the invertibility of $I - F_y(y, q)$ at the relaxed minimizing point.

We discuss necessary conditions for minimum in §2 (with proof in §5) and then consider, in §3, 4, 6, and 7, the special case of a control problem defined by a Uryson-type integral equation. We study the existence aspects of the control problem for integral equations assuming $\mathscr{Y}$ to be $L^1(T, E_n)$; and we examine necessary conditions for a relaxed minimum assuming that $\mathscr{Y}$ is either $L^p(T, E_n)$ for $1 < p < \infty$ or $C(T, E_n)$. Necessary conditions for an original minimum will be discussed separately along the lines of [*6*]. We might mention, finally, that certain more general unilateral and minimax control problems that have been investigated for ordinary differential equations [*7–9*] extend quite naturally to integral equations; they will also be discussed elsewhere.

I wish to acknowledge with thanks several stimulating conversations with J. Frampton.

2. The General Control Problem

We use the term "derivative" in the Frechet sense but relative to a set: specifically, if $\mathscr{X}$ and $\mathscr{Z}$ are Banach spaces, $x_1 \in \Gamma \subset \mathscr{X}$ and $h : \Gamma \to \mathscr{Z}$ then we say that a linear operator $H : \mathscr{X} \to \mathscr{Z}$ is a derivative of h at x_1 if $| h(\gamma) - h(x_1) - H(\gamma - x_1)| = o(| \gamma - x_1 |)$ for all $\gamma \in \Gamma$. We use the notation $h_x(x_1, y_1)$, $h_y(x_1, y_1)$ to denote partial derivatives if h is a function of two arguments. The symbol I represents the identity operator on $\mathscr{Y}$. If Q is a convex subset of a linear space, $\mathscr{X}$ is any set, $(x_1, q_1) \in \mathscr{X} \times Q$, $q \in Q$, and h is a function from $\mathscr{X} \times Q$ to some Banach space, we write $Dh(x_1, q_1; q - q_1)$ for the derivative at 0 of the function $t \to h(x_1, q_1 + t(q - q_1))$ on $[0, 1]$.

Theorem 2.1 below is patterned after [*6*, Theorem 2.2, p. 644] and relies ultimately on a basic construction of McShane [*10*, pp. 17–18].

Theorem 2.1. *Let $\mathscr{Y}$ be a Banach space, Q a convex subset of a linear space, $\omega^\square$ an array with real elements $\omega^{ij}(i, j = 1,..., m)$ considered as an element of E_{m^2} with origin $0^\square$,*

$$\Omega = \left\{\omega^\square \mid \omega^{ij} \geqslant 0, \sum_{i,j=1}^{m} \omega^{ij} \leqslant 1\right\},$$

and $(\bar{y}, \bar{q})$ a relaxed minimizing point. Assume, furthermore, that for each fixed subset $\{q_{ij} \mid i, j = 1,..., m\}$ of Q there exists a neighborhood $\tilde{\mathscr{Y}} \times \tilde{\Omega}$ of $(\bar{y}, 0^\square)$ in $\mathscr{Y} \times \Omega$ such that the functions

$$(y, \omega^\square) \to F\left(y, \bar{q} + \sum_{i,j=1}^{m} \omega^{ij}(q_{ij} - \bar{q})\right) : \tilde{\mathscr{Y}} \times \tilde{\Omega} \to \mathscr{Y}$$

and

$$(y, \omega^\square) \to c\left(y, \bar{q} + \sum_{i,j=1}^{m} \omega^{ij}(q_{ij} - \bar{q})\right) : \tilde{\mathscr{Y}} \times \tilde{\Omega} \to E_m$$

are continuous, have derivatives at $(\bar{y}, 0^\square)$ (relative to $\mathscr{Y} \times \tilde{\Omega}$) and continuous partial derivatives with respect to y on $\tilde{\mathscr{Y}} \times \tilde{\Omega}$, and that the operator $I - F_y(\bar{y}, \bar{q})$ is a linear homeomorphism of $\mathscr{Y}$ onto $\mathscr{Y}$. Then there exist $\gamma \geqslant 0$ and $\lambda \in E_m$ such that $\gamma + | \lambda | \neq 0$,

$$(2.1.1.) \quad \lambda \cdot \{c_y(\bar{y}, \bar{q})(I - F_y(\bar{y}, \bar{q}))^{-1} DF(\bar{y}, \bar{q}; q - \bar{q}) + Dc(\bar{y}, \bar{q}; q - \bar{q})\} \geqslant 0$$

for all $q \in Q$, *and*

$$(2.1.2)\quad (\gamma\,\delta_1 - \lambda) \cdot c(\bar{y}, \bar{q}) = \operatorname*{Min}_{\xi \in B_1} (\gamma\,\delta_1 - \lambda)\,\xi,$$

where $\delta_1 = (1, 0, \ldots, 0) \in E_m$.

3. Control Problems Defined by Uryson-Type Integral Equations. Existence of Relaxed and Approximate Minimizing Solutions

Let T, R, and B be compact metric spaces. We assume that a nonnegative, finite, regular, complete, and nonatomic measure dt is defined on the Lebesgue extension of the Borel field of sets in T and we consider the corresponding product measure $dt\,d\tau$ on $T \times T$. The symbol $|M|$ represents the measure of $M \subset T$, $\int h(t)\,dt$ the integral over T, $|a, b|$ the distance in a metric space, and $|a|$ (or $|a|_E$) the norm in a normed linear space E. We represent by $L^p(T, \mathscr{X})$ $(1 \leqslant p < \infty)$ the Banach space of measurable functions h from T to a Banach space $\mathscr{X}$ with the norm $|h(\cdot)|_p = \{\int |h(t)|_{\mathscr{X}}^p\,dt\}^{1/p}$ and by $C(T, \mathscr{X})$ the Banach space of continuous h from T to $\mathscr{X}$ with the norm $|h(\cdot)|_\infty = \sup_{t \in T} |h(t)|_{\mathscr{X}}$. We also set $L^p(T) = L^p(T, E_1)$ and $C(T) = C(T, E_1)$.

Original and Relaxed Controls.

Let $\mathscr{R}$ be the class of measurable mappings from T to R. As in [5], we refer to functions from T to R as "original controls." Let S be the class of regular Borel probability measures on R, and $\mathscr{S}$ the class of "measurable relaxed controls," that is, mappings σ from T to S that are measurable in the sense that $t \to \int_R \phi(r)\,\sigma(dr; t)$ is measurable on T for every continuous $\phi : R \to E_1$. We define $\mathscr{R}$ as a subset of $\mathscr{S}$ by identifying the function $t \to \rho(t)$ with the function $t \to \sigma_\rho(t)$, where $\sigma_\rho(t)$ is a measure concentrated at $\rho(t)$ with mass 1. We also identify all controls, original or relaxed, that differ only on sets of measure 0.

Topology in the Space of Measurable Relaxed Controls.

We define a topology in $\mathscr{S}$ as in [5, p. 631]; we represent by $\mathscr{B}$ the Banach space (which is actually the space $L^1(T, C(R))$) of real-valued functions ϕ on $T \times R$, continuous on R for every t, measurable on T for every r, with $t \to \sup_{r \in R} |\phi(t, r)|$ integrable, and with

$|\phi|_{\mathscr{B}} = \int \sup_{r \in R} |\phi(t, r)|\, dt$. We then define every $\sigma \in \mathscr{S}$ as an element of $\mathscr{B}^*$ (the topological dual of $\mathscr{B}$) by setting

$$\langle \sigma, \phi \rangle = \int dt \int_R \phi(t, r)\, \sigma(dr; t) \qquad \text{for all} \quad \phi \in \mathscr{B}.$$

The topology we choose for $\mathscr{B}^*$, and its subsets $\mathscr{S}$ and $\mathscr{R}$, is the weak star topology in $\mathscr{B}^*$ (the $\mathscr{B}$ topology of $\mathscr{B}^*$). It follows that $\lim_{i \to \infty} \sigma_i = \sigma$ implies

$$\lim_{i \to \infty} \int dt \int_R \phi(t, r)\, \sigma_i(dr; t) = \int dt \int_R \phi(t, r)\, \sigma(dr; t)$$

for every $\phi \in \mathscr{B}$.

Formulation of the Uryson-Type Control Problem

Now let $g = (g^1, \dots, g^n)$, and let $(t, \tau, v, r, b) \to g(t, \tau, v, r, b) : T \times T \times E_n \times R \times B \to E_n$ be measurable in (t, τ) for every fixed (v, r, b) and continuous in (v, r, b) for every fixed (t, τ). We also assume that $g^i(t, \tau, v, r, b) = g^i(\tau, v, r, b) (i = 1, \dots, m \leqslant n)$ is independent of t for all (τ, v, r, b). Let

$$f(t, \tau, v, s, b) = \int_R g(t, \tau, v, r, b)\, s(dr)$$

for all (t, τ, v, b) and all $s \in S$. We consider solutions (y, σ, b) of the integral equation

$$y(t) = \int f(t, \tau, y(\tau), \sigma(\tau), b)\, d\tau \qquad (t \in T)$$

in $\mathscr{Y} \times \mathscr{S} \times \mathscr{B}$, where $\mathscr{Y}$ is some Banach space of measurable functions from T to E_n.

A solution (y, σ, b) is "a relaxed admissible solution" if $(y^1, y^2, \dots, y^m) \in B_1$ (observe that $y^i (i = 1, \dots, m)$ are constant on T since $g^i(t, \tau, v, r, b) (i = 1, \dots, m)$ are independent of t). A relaxed admissible solution $(\bar{y}, \bar{\sigma}, \bar{b})$ is "a relaxed minimizing solution" if $\bar{y}^1 \leqslant y^1$ for all relaxed admissible solutions (y, σ, b).

We relate the control problem just described to the general problem discussed in §2 in the following manner: Let $\mathscr{U} = \mathscr{R} \times B$ and $Q = \mathscr{S} \times B$. We let the mappings $(y, q) \to F(y, q) = F(y, \sigma, b)$ and $(y, q) \to c(y, q) = c(y, \sigma, b)$ be defined, for $q = (\sigma, b) \in \mathscr{S} \times B$ and $y \in \mathscr{Y}$, by

$$F(y, \sigma, b)(t) = \int f(t, \tau, y(\tau), \sigma(\tau), b)\, d\tau \qquad (t \in T),$$

$$c^i(y, \sigma, b) = \int f^i(\tau, y(\tau), \sigma(\tau), b)\, d\tau \qquad (i = 1, \dots, m),$$

if this defines $F(y, \sigma, b)$ as an element of $\mathscr{Y}$. Otherwise we set, for some $a \in \mathscr{Y}$, $a \neq 0$,

$$F(y, \sigma, b) = y + a,$$

$$c(y, \sigma, b) = 0.$$

We can easily verify that, in the case where T is the interval $[t_0, t_1]$ of the real axis and $g(t, \tau, v, r, b)$ has a special form, the Uryson integral equation becomes an ordinary differential equation, our control problem the "standard" control problem, and the results that follow are similar to previous results [*5*, Theorem 3.1, p. 633], [*6*, Theorem 3.4, p. 648]. We further discuss this problem in §4.

We can now state existence and approximation theorems that we prove in §5. In both of these theorems we set $\mathscr{Y} = L^1(T, E_n)$.

THEOREM 3.1. *There exists a relaxed minimizing solution if the following conditions are satisfied*:

(3.1.1) *the class of relaxed admissible solutions is nonempty for* $\mathscr{Y} = L^1(T, E_n)$ *and*

(3.1.2) *there exists a positive function* ψ, *integrable on* $T \times T$ *and such that, for every solution* $(\tilde{y}, \tilde{\sigma}, \tilde{b})$ *of the equation* $y = F(y, \sigma, b)$, *we have*

$$|g(t, \tau, \tilde{y}(\tau), r, b)| \leqslant \psi(t, \tau) \qquad \text{on} \qquad T \times T \times R \times B.$$

THEOREM 3.2. *Let* $\bar{y}$ *be the unique solution of the equation* $y = F(y, \sigma, b)$ *for* $\sigma = \bar{\sigma} \in \mathscr{S}$ *and* $b = \bar{b} \in B$. *Assume, furthermore, that*

(3.2.1) *the equation* $y = F(y, \rho, \bar{b})$ *admits at least one solution* y *in* $L^1(T, E_n)$ *for each* $\rho \in \mathscr{R}$ *in some neighborhood of* $\bar{\sigma}$ *and condition* (3.1.2) *is satisfied.*

Then there exists a sequence $\{\rho_i\}_{i=1}^{\infty}$ *in* $\mathscr{R}$ *and a sequence* $\{y_i\}_{i=1}^{\infty}$ *in* $L^1(T, E_n)$ *such that* $y_i = F(y_i, \rho_i, \bar{b})$ *and*

$$\lim_{i\to\infty} c(y_i, \rho_i, \bar{b}) = c(\bar{y}, \bar{\sigma}, \bar{b}).$$

4. NECESSARY CONDITIONS FOR A RELAXED MINIMUM OF A URYSON-TYPE CONTROL PROBLEM

We shall investigate necessary conditions for a point $(\bar{y}, \bar{\sigma}, \bar{b})$ to be a relaxed minimizing solution in a more restricted framework than was required in §3.

Assumption 4.1.

(4.1.1) $\mathscr{Y}$ is either $L^p(T, E_n)$ for $1 < p < \infty$ or $C(T, E_n)$, and T and R have the properties described in §3;

(4.1.2) B is a compact and convex subset of a Banach space;

(4.1.3) the function g has a derivative with respect to (v, b), and g, g_v, and g_b are measurable in (t, τ) for every (v, r, b) and continuous in (v, r, b) for every (t, τ);

(4.1.4) if $\mathscr{Y} = L^p(T, E_n)$ then there exist measurable positive functions ψ_0 and ψ_1 on $T \times T$ and numbers α and β such that

$$0 \leqslant \alpha < p - 1, \qquad 0 \leqslant \beta < p, \qquad \int | \psi_0(\cdot, \tau)|_p^{p/(p-\beta)} \, d\tau < \infty,$$

$$\int | \psi_1(t, \cdot)|_{p/(p-1-\alpha)}^p \, dt < \infty, \qquad \int | \psi_1(\cdot, \tau)|_p^{p/(p-1-\alpha)} \, d\tau < \infty,$$

and, for all $(t, \tau, v, r, b) \in T \times T \times E_n \times R \times B$,

$$| \hat{g}(t, \tau, v, r, b)| \leqslant (1 + | v |^\beta) \psi_0(t, \tau) \qquad \text{for} \quad \hat{g} = g \quad \text{or} \quad g_b$$

and

$$| g_v(t, \tau, v, r, b)| \leqslant (1 + | v |^\alpha) \psi_1(t, \tau);$$

if $\mathscr{Y} = C(T, E_n)$ then there exists a compact set D in E_n containing $\{ \bar{y}(t) | \, t \in T\}$ in its interior a measurable ψ_0 on $T \times T$, and a modulus of continuity Φ such that, for $\hat{g} = g, g_v$ and g_b,

$$| \hat{g}(t, \tau, v, r, b)| \leqslant \psi_0(t, \tau) \quad \text{on} \quad T \times T \times D \times R \times B, \ \operatorname*{Sup}_{t \in T} \int \psi_0(t, \tau) \, dt < \infty,$$

and

$$\int \operatorname*{Sup}_{D \times R \times B} | \hat{g}(t_1, \tau, v, r, b) - \hat{g}(t_2, \tau, v, r, b)| \, d\tau \leqslant \Phi(| t_1, t_2 |) \ \ (t_1, t_2 \in T);$$

furthermore, for each $(\tau, r, b) \in T \times R \times B$, the function $v \to g_v(t, \tau, r, b)$ is continuous on D uniformly in $t \in T$;

(4.1.5) for $k(t, \tau) = f_v(t, \tau, \bar{y}(\tau), \bar{\sigma}(\tau), \bar{b})$ on $T \times T$, the integral equation

$$w(t) = \int k(t, \tau) \, w(\tau) \, d\tau \qquad (t \in T)$$

has only the solution $w(\cdot) = 0$ in $\mathscr{Y}$.

Resolvent Kernel

If there exists a measurable real matrix-valued function

$$k^* = (k_j^{*i}) \qquad (i, j = 1, \ldots, n)$$

on $T \times T$ such that, for every $x \in \mathscr{Y}$, the relations

$$w(t) = \int k(t, \tau)\, w(\tau)\, d\tau + x(t) \qquad \text{a.e. in } T$$

and

$$w(t) = \int k^*(t, \tau)\, x(\tau)\, d\tau + x(t) \qquad \text{a.e. in } T$$

are equivalent in $\mathscr{Y}$, we refer to k^* as a resolvent kernel of k.

We can now state necessary conditions for a relaxed minimum in Theorem 4.2 below. Conditions (1), (2), and (3) of (4.2.2) are generalizations of respectively, the Weierstrass E-condition, transversality with respect to parameters and initial conditions, and transversality with respect to the end conditions of the calculus of variations.

THEOREM 4.2. *Let* $(\bar{y}, \bar{\sigma}, \bar{b})$ *be a relaxed minimizing solution, and let Assumption* 4.1 *be satisfied. Then*

(4.2.1) *there exists a resolvent kernel* k^* *of* k *such that* k_j^{*i} *is independent of* t *for* $i = 1, 2, \ldots, m$; *and*

$$\int |\, k^*(\cdot, \tau)|_p^{p/(p-1)}\, d\tau < \infty \qquad \textit{if} \quad \mathscr{Y} = L^p(T, E_n)$$

and

$$\operatorname*{Sup}_{t \in T} \int |\, k^*(t, \tau)|\, d\tau < \infty \qquad \textit{if} \quad \mathscr{Y} = C(T, E_n);$$

(4.2.2) *there exist* $\lambda = (\lambda^1, \ldots, \lambda^m) \in E_m$ *and* $\gamma \geqslant 0$, *with* $\gamma + |\lambda| \neq 0$, *and such that, setting*

$$\hat{\lambda} = (\lambda^1, \ldots, \lambda^m, 0, \ldots, 0) = (\lambda, 0, \ldots, 0) \in E_n\,,$$

$$\zeta^j(\tau) = \sum_{i=1}^{m} \lambda^i k_j^{*i}(\tau) + \frac{\hat{\lambda}^j}{|\, T\,|} \qquad (\tau \in T, j = 1, \ldots, n),$$

$$\zeta(\tau) = (\zeta^1(\tau), \ldots, \zeta^n(\tau)) \qquad (\tau \in T),$$

$$H_1(s, \theta) = \int \zeta(\tau) \cdot f(\tau, \theta, \bar{y}(\theta), s, \bar{b})\, d\tau \qquad ((s, \theta) \in S \times T),$$

and

$$H_2(b) = \int_{T \times T} \zeta(\tau) \cdot Df(\tau, \theta, \bar{y}(\theta), \bar{\sigma}(\theta), \bar{b}; b - \bar{b})\, d\tau d\theta \qquad (b \in B),$$

the following conditions are satisfied:

(*The Weierstrass E-Condition*)

$$(1)\quad H_1(\bar{\sigma}(\theta), \theta) = \operatorname*{Min}_{s \in S} H_1(s, \theta)$$

$$= \operatorname*{Min}_{r \in R} \int \zeta(\tau) \cdot g(\tau, \theta, \bar{y}(\theta), r, \bar{b})\, d\tau \quad \textit{for almost all} \quad \theta \in T,$$

(*Transversality Conditions*)

$$(2)\quad \operatorname*{Min}_{b \in B} H_2(b) = 0,$$

and

$$(3)\quad (\gamma\delta_1 - \lambda) \cdot (\bar{y}^1, ..., \bar{y}^m) = \operatorname*{Min}_{\xi \in B_1} (\gamma\delta_1 - \lambda)\, \xi,$$

$$\textit{where} \qquad \delta_1 = (1, 0, ..., 0) \in E_m .$$

An Illustration

As an illustration, we shall apply Theorem 4.2 to the following "standard" relaxed control problem: With R, $\mathscr{R}$ and $\mathscr{S}$ defined as before, let T be the closed interval $[t_0, t_1]$ of the real axis, $d\tau$ the Lebesgue measure on T, B_1 and B convex subsets of E_m and E_ℓ, respectively, ϕ_0 a continuously differentiable mapping from B into E_m with the image B_0, and h: $T \times E_m \times R \to E_m$. We wish to determine functions $\bar{x} : T \to E_m$ and $\bar{\sigma} : T \to S$ that yield the minimum of $x^1(t_1)$ among all absolutely continuous x and all $\sigma \in \mathscr{S}$ that satisfy the relations:

$$\frac{dx(\tau)}{d\tau} = \int_R h(\tau, x(\tau), r)\, \sigma(dr; \tau) \qquad \text{a.e. in } T,$$

$$x(t_0) \in B_0, \qquad x(t_1) \in B_1 .$$

We set $n = 2m, g = (g^1, ..., g^n)$, $y = (y^1, ..., y^n)$, $v = (w_1, w_2)$ with $w_1, w_2 \in E_m$, and, for all $(t, \tau, v, r, b) \in T \times T \times E_n \times R \times B$ and $i = 1, 2, ..., m$,

$$y^{i+m}(t) = x^i(t),$$

$$y^i(t) = x^i(t_1),$$

$$g^{i+m}(t, \tau, v, r, b) = \begin{cases} h^i(\tau, w_2, r) + \phi_0{}^i(b)/(t_1 - t_0) & \text{for} \quad \tau \leqslant t, \\ \phi_0{}^i(b)/(t_1 - t_0) & \text{for} \quad \tau > t, \end{cases}$$

$$g^i(t, \tau, v, r, b) = g^{i+m}(t\ \ , \tau, v, r, b).$$

We then observe that our new problem is formally equivalent to the Uryson-type control problem considered in §3 and in the present section. We can easily verify that Theorem 4.2 is applicable if we set $\mathscr{Y} = C(T, E_n)$ and assume that

$$(\tau, w, r) \to h(\tau, w, r) \qquad \text{and} \qquad h_w(\tau, w, r)$$

exist on $T \times E_m \times R$, are continuous in (w, r) and measurable in τ, and that $| h(\tau, w, r)|$ and $| h_w(\tau, w, r)|$ are bounded by an integrable function of τ for all $(w, r) \in D \times R$, where D is some compact set in E_m containing the trajectory $\{\bar{x}(t) | \, t \in T\}$ in its interior.

We can evaluate the resolvent kernel k^* by a straightforward (if somewhat tedious) computation and determine that

$$\zeta^j(\tau) = \frac{1}{t_1 - t_0} \lambda^j = \frac{1}{t_1 - t_0} z^j(t_1) \qquad (\tau \in T, j = 1,..., m),$$

$$\zeta^{j+m}(\tau) = -dz^j(\tau)/d\tau \qquad (\tau \in T, j = 1,..., m),$$

where the absolutely continuous function $\tau \to z(\tau) : T \to E_m$ is the solution of the system

$$\frac{dz(\tau)}{d\tau} = \dot{z}(\tau) = -A^T(\tau)\, z(\tau) \qquad \text{a.e. in } T,$$

$$z(t_1) = \lambda,$$

A^T is the transpose of A, and $A(\tau) = \int_R h_w(\tau, \bar{x}(\tau), r)\, \bar{\sigma}(dr; \tau)(\tau \in T)$. It follows then that

$$H_1(\bar{\sigma}(\theta), r) = z(\theta) \cdot \int_R h(\theta, \bar{x}(\theta), r)\, \bar{\sigma}(dr; \theta) + \frac{1}{| T |} z(t_0) \cdot \bar{x}(t_0) \ (\theta \in T, r \in R).$$

Thus relation (1) of Theorem 4.2 yields the familiar Weierstrass E-condition (maximum principle) for the relaxed control problem defined by ordinary differential equations. In a similar manner, relations (2) and (3) yield the support (transversality) conditions at the initial and terminal points t_0 and t_1, respectively.

5. Proof of Theorem 2.1

We first consider the special case where

$$B_1 = \{(b^1,..., b^m) \mid b^i = 0 \ (i = 2,..., m)\}.$$

Consider the equation

$$y = F\left(y, \bar{q} + \sum_{i,j=1}^{m} \omega^{ij}(q_{ij} - \bar{q})\right) \tag{5.1}$$

for an arbitrary choice of $q^\square = (q_{ij})$ with $q_{ij} \in Q$. We can apply, with minor changes, the proof of the implicit function theorem [*11*, p. 265] to show that there exists a neighborhood $\tilde{Y} \times \tilde{\Omega}$ of $(\bar{y}, 0^\square)$ relative to $\mathscr{Y} \times \Omega$ such that Eq. (5.1) has a unique solution $y = \eta(\omega^\square, q^\square) \in \tilde{Y}$ for every $\omega^\square \in \tilde{\Omega}$ and the function $\omega^\square \to \eta(\omega^\square, q^\square) : \tilde{\Omega} \to \tilde{Y}$ is continuous and has a derivative at $0^\square$. It follows that

$$\omega^\square \to \tilde{c}(\omega^\square, q^\square) = c\left(\eta(\omega^\square, q^\square), \bar{q} + \sum_{i,j=1}^{m} \omega^{ij}(q_{ij} - \bar{q})\right) : \tilde{\Omega} \to E_m$$

is also continuous and has a derivative at $0^\square$.

Now let $\theta^{11} = \theta, \theta^{ij} = 0 \ \ ((i,j) \neq (1,1)), \tilde{q}_{ij} = q (i,j = 1,..., m)$, $h(\theta; q) = \tilde{c}(\theta^\square, \tilde{q}^\square)$, $V = \{dh(0; q)/d\theta \mid q \in Q\}$, and let W be the convex cone in E_m generated by V; that is,

$$W = \left\{\sum_{i=1}^{m} a^i v_i \mid v_i \in V, a^i \geqslant 0\right\}.$$

We shall show in the sequel that there exists $\lambda \in E_m$ such that $|\lambda| \neq 0$, $\lambda^1 \geqslant 0$, and $\lambda \cdot w \geqslant 0$ for all $w \in W$.

If this last statement is true, then $\lambda \cdot dh(0; q)/d\theta \geqslant 0$ for all $q \in Q$. We have $dh(0; q)/d\theta = c_y(\bar{y}, \bar{q})\, \eta_{\omega^{11}}(0^\square, \tilde{q}^\square) + Dc(\bar{y}, \bar{q}; q - \bar{q})$. Also, the differentiation of both sides of the equation

$$\eta(\theta^\square; \tilde{q}^\square) = F(\eta(\theta^\square; \tilde{q}^\square), \bar{q} + \theta^{11}(q - \bar{q}))$$

with respect to θ^{11} at $0^\square$ yields

$$\eta_{\omega^{11}}(0^\square, \tilde{q}^\square) = F_y(\bar{y}, \bar{q})\, \eta_{\omega^{11}}(0^\square, \tilde{q}^\square) + DF(\bar{y}, \bar{q}; q - \bar{q}).$$

We then conclude that

$$\begin{aligned} \lambda \cdot dh(0; q)/d\theta &= \lambda \cdot \{c_y(\bar{y}, \bar{q})(I - F_y(\bar{y}, \bar{q}))^{-1} DF(\bar{y}, \bar{q}; q - \bar{q}) \\ &\quad + Dc(\bar{y}, \bar{q}; q - \bar{q})\} \geqslant 0 \qquad \text{for all} \quad q \in Q. \end{aligned} \tag{5.2}$$

We now proceed to prove that there exists a point λ as just described. Indeed, assume the contrary. Then it follows from elementary properties of convex sets that there exists a point $w = (w^1, 0,..., 0)$

in the interior of W, linearly independent $w_i \in W$ and positive $\alpha^i(i = 1,..., m)$ such that

$$w^1 < 0 \qquad \text{and} \qquad w = \sum_{i=1}^{m} \alpha^i w_i .$$

By the definition of W, there exist points q_{ij} and numbers $a^{ij}(i, j = 1,..., m)$ such that $a^{ij} \geqslant 0$ and

$$w_i = \sum_{j=1}^{m} a^{ij}\, dh(0; q_{ij})/d\theta \qquad (i = 1,..., m).$$

Since the w_i are independent, the matrix $(w_i^j)(i, j = 1,..., m)$ is nonsingular.

Now let $\tilde{\Gamma} = \{\gamma \in E_m \mid 0 \leqslant \gamma^i \leqslant \gamma_{\max}(i = 1,..., m)\}$, where $\gamma_{\max}$ is positive and sufficiently small so that $\omega^{\square}(\gamma) = (\omega^{ij}(\gamma)) = (\gamma^i a^{ij}) \in \tilde{\Omega}$, and consider the function $\gamma \to k(\gamma) = \tilde{c}(\omega^{\square}(\gamma); q^{\square}) : \tilde{\Gamma} \to E_m$. This function is continuous and has a derivative

$$k_\gamma(0) = (\partial k(0)/\partial\gamma^1,..., \partial k(0)/\partial\gamma^m)$$

at 0 relative to $\tilde{\Gamma}$ (where $\partial k(0)/\partial\gamma^i$ are right-hand derivatives). Furthermore,

$$\begin{aligned}\partial k(0)/\partial\gamma^l &= \sum_{i,j=1}^{m} dh(0; q_{ij})/d\theta \cdot \partial\omega^{ij}(0)/\partial\gamma^l \\ &= \sum_{j=1}^{m} dh(0; q_{lj})/d\theta \cdot a^{lj} = w_l \qquad (l = 1,..., m);\end{aligned}$$

hence the derivative $k_\gamma(0) = (w_i^j)$ has an inverse and

$$k_\gamma(0)\,\alpha = w = (w^1, 0,..., 0) \qquad \text{for} \quad \alpha = (\alpha^1,..., \alpha^m).$$

It follows (as in [*6*, p. 650]) that there exists a solution $\epsilon \to \gamma(\epsilon)$ of the equation

$$k(\gamma(\epsilon)) - k(\gamma(0)) = \epsilon w$$

for all sufficiently small positive ϵ, and $\gamma^i(\epsilon) \to_{\epsilon \to +0} +0\ (i = 1,..., m)$. There exist, therefore, $q = \bar{q} + \sum_{i,j=1}^{m} \omega^{ij}(\gamma(\epsilon))(q_{ij} - \bar{q}) \in \mathcal{Q}$ and $y = \eta(\omega^{\square}(\gamma(\epsilon)); q^{\square}) \in \mathcal{Y}$ such that $y = F(y, q)$, $c^i(y, q) = 0$ $(i = 2,..., m)$, and $c^1(y, q) < c^1(\bar{y}, \bar{q})$, contradicting the assumption that $(\bar{y}, \bar{q})$ is a relaxed minimizing point.

We conclude that when $B_1 = \{(b^1,..., b^m) | \, b^i = 0(i = 2,..., m)\}$ there exists a point $\lambda \in E_m$ such that $|\lambda| \neq 0$, $\lambda^1 \geqslant 0$, and relation (5.2) is satisfied.

We now consider the general case and define the sets $Q^{\#}$ and $B_1{}^{\#}$ and the functions $F^{\#} : \mathscr{Y} \times Q^{\#} \to \mathscr{Y}$ and $c^{\#} : \mathscr{Y} \times Q^{\#} \to E_{m+1}$ by

$$Q^{\#} = Q \times B_1, \qquad B_1{}^{\#} = \{(v^0, v^1, ..., v^m) \mid v^i = 0 (i = 1, ..., m)\},$$

$$F^{\#}(y, q, \xi) = F(y, q),$$

$$c^{\#0}(y, q, \xi) = c^1(y, q),$$

$$c^{\#l}(y, q, \xi) = c^l(y, q) - \xi^l \qquad (l = 1, ..., m).$$

Clearly, the point $(\bar{y}, \bar{q}, \bar{\xi})$ is a relaxed minimizing point for the problem defined by $\mathscr{Y}, Q^{\#}, B_1{}^{\#}, F^{\#}$ and $c^{\#}$, which is of the form just investigated. The conclusions of the theorem follow from relation (5.2) applied to the transformed problem; the details of the argument are as in [*6*, Proof of Theorem 2.2, p. 650]. Q.E.D.

6. Existence of Relaxed and Approximate Minimizing Solutions for Uryson-type Problems. Proofs.

Lemma 6.1. *Let condition* (3.1.2) *be satisfied and let*

$$Y_2 = \{y \in \mathscr{Y} \mid y = F(y, \sigma, b), \sigma \in \mathscr{S}, b \in B\}.$$

Then every sequence $\{y_i\}_{i=1}^{\infty}$ *in* Y_2 *has a subsequence converging to some* $\tilde{y} \in \mathscr{Y} = L^1(T, E_n)$.

Proof: Let $y_i = F(y_i, \sigma_i, b_i) (i = 1, 2, ...)$, $y_i \in \mathscr{Y}$, $\sigma_i \in \mathscr{S}$, and $b_i \in B$. We must show that there exist a sequence J of natural numbers and a point $\tilde{y}$ in $\mathscr{Y}$ such that $\lim_{i \in J} y_i = \tilde{y}$ in $\mathscr{Y}$.

Let, for $v \in E_n$, $\chi(v) = 1$ if $|v| \leqslant 1$ and $\chi(v) = 1/(|v|^2)$ if $|v| > 1$,

$$\tilde{g}(t, \tau, v, r, b) = \chi(g(t, \tau, v, r, b)/\psi(t, \tau))\, g(t, \tau, v, r, b),$$

and

$$\tilde{f}(t, \tau, v, s, b) = \int_R \tilde{g}(t, \tau, v, r, b)\, s(dr),$$

for all t, τ, v, r, b and all $s \in S$. Then, by (3.1.2),

$$\tilde{g}(t, \tau, y(\tau), r, b) = g(t, \tau, y(\tau), r, b)$$

on $T \times T \times R \times B$ for every $y \in Y_2$; hence every solution (y, σ, b) of the equation $y = F(y, \sigma, b)$ also satisfies the equation

$$y(t) = \int \tilde{f}(t, \tau, y(\tau), \sigma(\tau), b)\, d\tau \qquad (t \in T). \tag{6.1.1}$$

Furthermore,

$$|\tilde{g}(t, \tau, v, r, b)| \leqslant \psi(t, \tau) \quad \text{on} \quad T \times T \times E_n \times R \times B$$

and $\tilde{g}$ is continuous in (v, r, b) and measurable in (t, τ).

Now let $\tilde{\psi}(t) = \int \psi(t, \tau)\, d\tau$ on T, $S_N = \{v \in E_n \mid |v| \leqslant N\}$, $P_N = \{t \in T \mid \tilde{\psi}(t) \leqslant N\}$, and $\epsilon > 0$. Then there exists $N = N(\epsilon)$ such that, for $P = P_{N(\epsilon)}$,

$$\int dt \int_{T-P} \psi(t, \tau)\, d\tau \leqslant \tfrac{1}{8}\epsilon. \tag{6.1.2}$$

Since $\tilde{g}$ is measurable in (t, τ), continuous in (v, r, b) on the compact set $S_N \times R \times B$, and $|\tilde{g}(t, \tau, v, r, b)| \leqslant \psi(t, \tau)$, the restriction of $\tilde{g}^i$ to $T \times P \times S_N \times R \times B$ is, for each $i = 1,\ldots, n$, an element of $L^1(T \times P,\ C(S_N \times R \times B))$; there exist, therefore, an integer $k = k(\epsilon)$ and functions $\alpha_j \in L^1(T \times P, E_n)$ and $\beta_j \in C(S_N \times R \times B)$ such that

$$\int_T \int_P \operatorname*{Max}_{S_N \times R \times B} \left| \tilde{g}(t, \tau, v, r, b) - \sum_{j=1}^{k} \beta_j(v, r, b)\, \alpha_j(t, \tau) \right| dt\, d\tau \leqslant \tfrac{1}{4}\epsilon. \tag{6.1.3}$$

Furthermore, each $\alpha_j \in L^1(T \times P, E_n)$ can be approximated by a finite sum $\sum_l b_l(\tau)\, a_l(t)$, where b_l are measurable characteristic functions on P and $a_l \in L^1(T, E_n)$. We conclude, therefore, that relation (6.1.3) can be rewritten, by appropriately changing the definitions of k and β_j, as

$$\int_T \int_P \operatorname*{Max}_{S_N \times R \times B} \left| \tilde{g}(t, \tau, v, r, b) - \sum_{j=1}^{k} \beta_j(v, r, b)\, b_j(\tau)\, a_j(t) \right| dt\, d\tau \leqslant \tfrac{1}{4}\epsilon, \tag{6.1.4}$$

and we may also assume that $|\beta_j(v, r, b)| \leqslant 1$ on $S_N \times R \times B$.

Now let

$$\gamma_{ji} = \gamma_{ji}(\epsilon) = \int_P b_j(\tau)\, d\tau \int_R \beta_j(y_i(\tau), r, b_i)\, \sigma_i(dr; \tau).$$

We observe that

$$|y_i(t)| \leqslant \int |\tilde{f}(t, \tau, y_i(\tau), \sigma_i(\tau), b_i)|\, d\tau \leqslant \int \psi(t, \tau)\, d\tau \leqslant N = N(\epsilon)$$

for $t \in P = P(\epsilon)$ and all $i = 1, 2, 3,\ldots$. Therefore, for all integers

p and q, for all $t \in T$, and for $k = k(\epsilon)$,

$$\begin{aligned} |y_p(t) - y_q(t)| &\leqslant 2 \int_{T-P} \psi(t, \tau)\, d\tau + \left| \int_P d\tau \left\{ \int_R \tilde{g}(t, \tau, y_p(\tau), r, b_p)\, \sigma_p(dr; \tau) \right.\right. \\ &\qquad \left.\left. - \int_R \tilde{g}(t, \tau, y_q(\tau), r, b_q)\, \sigma_q(dr; \tau) \right\} \right| \\ &\leqslant 2 \int_{T-P} \psi(t, \tau)\, d\tau + \left| \sum_{j=1}^{k} (\gamma_{jp} - \gamma_{jq})\, a_j(t) \right| \\ &\qquad + \sum_{i=p, i=q} \int_P \operatorname*{Max}_{r \in R} \left| \tilde{g}(t, \tau, y_i(\tau), r, b_i) \right. \\ &\qquad \left. - \sum_{j=1}^{k} \beta_j(y_i(\tau), r, b_i)\, b_j(\tau)\, a_j(\tau) \right| d\tau; \end{aligned}$$

hence, in view of relations (6.1.2) and (6.1.4),

$$\int_T |y_p(t) - y_q(t)|\, dt \leqslant \tfrac{1}{4}\epsilon + \sum_{j=1}^{k} |\gamma_{jp} - \gamma_{jq}| \int_T |a_j(t)|\, dt + \tfrac{1}{2}\epsilon. \qquad (6.1.5)$$

Given any infinite subsequence $\tilde{J}$ of $\{1, 2,...\}$, we can determine a subsequence $J' = J'(\tilde{J}, \epsilon)$ such that the sequences $\{\gamma_{ji}(\epsilon)\}_{i \in J'}$ have a limit, for each $j = 1, 2,..., k(\epsilon)$, since $|\gamma_{ji}| \leqslant |P| \leqslant |T|$ for all i and j. Now let $J_0 = \{1, 2,...\}$, $J_{l+1} = J'(J_l, 2^{-l})(l = 0, 1, 2,...)$, and let $\bar{J}$ be the diagonal subsequence of $J_0, J_1, ...$ Then $\{\gamma_{ji}(\epsilon)\}_{i \in \bar{J}}$ converges for each $\epsilon > 0$ and $j = 1, 2,..., k(\epsilon)$, and there exists an integer $i_0 = i_0(\epsilon)$ such that, in view of (6.1.5),

$$\int_T |y_p(t) - y_q(t)|\, dt \leqslant \epsilon$$

provided $p \geqslant q \geqslant i_0(\epsilon)$ and $p, q \in \bar{J}$.

We conclude that $\{y_i(\cdot)\}_{i \in \bar{J}}$ is a Cauchy sequence in $L^1(T, E_n)$ and converges, therefore, to some $\tilde{y}$ in $L^1(T, E_n)$. Q.E.D.

6.2. *Proof of Theorem 3.1.* Let $\{(y_i, \sigma_i, b_i)\}_{i=1}^{\infty}$ be a sequence in $\mathscr{Y} \times \mathscr{S} \times B$ and $y_i = F(y_i, \sigma_i, b_i)$. By [6, Theorem 2.5, p. 632] the set $\mathscr{S}$ is sequentially compact, and by Lemma 6.1 there exists a sequence J and a $\tilde{y} \in \mathscr{Y}$ such that $\lim_{i \in J} y_i = \tilde{y}$. We may choose J so that $\lim_{i \in J} \sigma_i = \tilde{\sigma}$ and $\lim_{i \in J} b_i = b$ for some $\tilde{\sigma} \in \mathscr{S}$ and $\tilde{b} \in B$, and $\lim_{i \in J} y_i = \tilde{y}$ a.e. in T, say for $t \in T'$.

For each fixed t and τ in T', $g(t, \tau, \cdot, \cdot, \cdot)$ is continuous, hence uniformly continuous, on the compact set $D_\tau \times R \times B$, where D_τ

is a compact subset of E_n containing $\tilde{y}(\tau)$ in its interior. It follows that $\lim_{i\in J} g(t, \tau, y_i(\tau), \cdot, b_i) = g(t, \tau, y(\tau), \cdot, \tilde{b})$ uniformly on R and

$$\lim_{i\in J} \int_R (g(t, \tau, y_i(\tau), r, b_i) - g(t, \tau, \tilde{y}(\tau), r, \tilde{b}))\, \sigma_i(dr; \tau) = \lim_{i\in J} \alpha_i(t, \tau) = 0$$

for all $t, \tau \in T'$. Furthermore, $|\alpha_i(t, \tau)| \leqslant 2\psi(t, \tau)$ on $T' \times T'$ and $\psi(t, \cdot)$ is integrable; therefore, for each $t \in T'$,

$$\begin{aligned}
\tilde{y}(t) &= \lim_{i\in J} y_i(t) \\
&= \lim_{i\in J} \int_T d\tau \int_R (g(t, \tau, y_i(\tau), r, b_i) - g(t, \tau, \tilde{y}(\tau), r, \tilde{b}))\, \sigma_i(dr; \tau) \\
&\quad + \lim_{i\in J} \int_T d\tau \int_R g(t, \tau, \tilde{y}(\tau), r, \tilde{b})\, \sigma_i(dr; \tau) \\
&= \lim_{i\in J} \int_T d\tau \int_R g(t, \tau, \tilde{y}(\tau), r, \tilde{b})\, \sigma_i(dr; \tau) \\
&= \int_T d\tau \int_R g(t, \tau, \tilde{y}(\tau), r, \tilde{b})\, \tilde{\sigma}(dr; \tau) = \int_T f(t, \tau, \tilde{y}(\tau), \tilde{\sigma}(t), \tilde{b})\, d\tau,
\end{aligned}$$

since the function $(\tau, r) \to g^j(t, \tau, \tilde{y}(\tau), r, \tilde{b}) \in \mathscr{B}$ (as defined in §3).

Thus $\tilde{y}(t) = F(\tilde{y}, \tilde{\sigma}, \tilde{b})(t)$ for $t \in T'$. By redefining $\tilde{y}$, if necessary, on $T - T'$, we can assert that $\tilde{y} = F(\tilde{y}, \tilde{\sigma}, \tilde{b})$ and thus the set of solutions of $y = F(y, \sigma, b)$ is nonempty and sequentially compact in $\mathscr{Y} \times \mathscr{S} \times B$. Since $y^j = c^j(y, \sigma, b)(j = 1,..., m)$ for every solution (y, σ, b), the $y^j(j = 1,..., m)$ are constant, and B_1 is closed, it follows that there exists a minimizing relaxed solution. Q.E.D.

6.3. *Proof of Theorem 3.2.* By [5, Theorem 2.4, p. 631], the set $\mathscr{R}$ is a dense subset of $\mathscr{S}$. There exists, therefore, a sequence $\{\rho_i\}_{i=1}^{\infty}$ in $\mathscr{R}$ converging to $\bar{\sigma}$. By (3.2.1), there exists an integer i_0 and a sequence $\{y_i\}_{i=i_0}^{\infty}$ in $\mathscr{Y}$ such that $y_i = F(y_i, \rho_i, \bar{b})$. It follows then, as in the proof of Theorem 3.1, that there exist $\tilde{y} \in \mathscr{Y}$ and a sequence J such that $\lim_{i\in J} y_i = \tilde{y}$, $\lim_{i\in J} \rho_i = \bar{\sigma}$, and $\tilde{y} = F(\tilde{y}, \bar{\sigma}, \bar{b})$. By the uniqueness assumption, $\tilde{y} = \bar{y}$. Thus

$$\lim_{i\in J} c^j(y_i, \rho_i, \bar{b}) = \lim y_i^j = \bar{y}^j \qquad (j = 1,..., m)$$

since the $y_i^j(j = 1,..., m)$ are constant. Q.E.D.

7. *Proof of Theorem 4.2.* We shall use the notation and the assumptions of Section 4. We also set, for a fixed choice of

$b_{ij}(i, j = 1,..., m)$ in B and $\sigma_{ij}(i, j = 1,..., m)$ in $\mathscr{S}$, and for all $\omega^\square \in \Omega$, $v \in E_n$, $y \in \mathscr{Y}$, and $t \in T$,

$$\tilde{b}(\omega^\square) = \bar{b} + \sum_{i,j=1}^{m} \omega^{ij}(b_{ij} - \bar{b}),$$

$$\tilde{\sigma}(\omega^\square) = \bar{\sigma} + \sum_{i,j=1}^{m} \omega^{ij}(\sigma_{ij} - \bar{\sigma}),$$

$$\tilde{f}(t, \tau, v, \omega^\square) = f(t, \tau, v, \sigma(\tau; \omega^\square), \tilde{b}(\omega^\square)),$$

$$\tilde{F}(y, \omega^\square)(t) = \int \tilde{f}(t, \tau, y(\tau), \omega^\square)\, d\tau.$$

LEMMA 7.1. *Let b_{ij} and $\sigma_{ij}(i, j = 1,..., m)$ be fixed. Then*

$$(y, \omega^\square) \to \tilde{F}(y, \omega^\square) : \mathscr{Y} \times \Omega \to \mathscr{Y}$$

is continuous in some neighborhood Γ of $(\bar{y}, 0^\square)$ and has a derivative at $(\bar{y}, 0^\square)$, the partial derivative $\tilde{F}_y$ exists and is continuous on Γ, and the following relations hold:

$$(\tilde{F}_y(y, \omega^\square)\, \Delta y)(t) = \int_T \tilde{f}_v(t, \tau, y(\tau), \omega^\square)\, \Delta y(\tau)\, d\tau \qquad (t \in T, y \in \mathscr{Y}, \Delta y \in \mathscr{Y}, \omega^\square \in \Omega),$$

$$\tilde{F}_{\omega^\square}(\bar{y}, 0^\square)(t) = \int_T \tilde{f}_{\omega^\square}(t, \tau, y(\tau), 0^\square)\, d\tau \qquad (t \in T, y \in \mathscr{Y}),$$

$$(\tilde{F}_y(\bar{y}, 0^\square)\, \Delta y)(t) = \int_T k(t, \tau)\, \Delta y(\tau)\, d\tau \qquad (t \in T, \Delta y \in \mathscr{Y}),$$

and

$$\tilde{F}_{\omega^{11}}(\bar{y}, 0^\square)(t) = \int_T f(t, \tau, \bar{y}(\tau), \sigma_{11}(\tau) - \bar{\sigma}(\tau), \bar{b})\, d\tau + \int_T Df(t, \tau, \bar{y}(\tau), \bar{\sigma}(\tau), \bar{b}; b_{11} - \bar{b})\, d\tau \qquad (t \in T).$$

Proof. We first consider the case $\mathscr{Y} = L^p(T, E_n)$ for $1 < p < \infty$. Let $y \in \mathscr{Y}$ and $\omega^\square \in \Omega$ be fixed. We observe that the function $(t, \tau) \to \tilde{f}(t, \tau, y(\tau), \omega^\square)$ is measurable on $T \times T$ and

$$\int |\tilde{f}(t, \tau, y(\tau), \omega^\square)|^p\, dt \leqslant \int |\psi_0(t, \tau)|^p\, (1 + |y(\tau)|^\beta)^p\, dt \leqslant |\psi_0(\cdot, \tau)|_p^p\, (1 + |y(\tau)|^\beta)^p < \infty$$

for almost all $\tau \in T$. Thus the function $t \to \tilde{f}(t, \tau, y(\tau), \omega^\square)$ belongs to $L^p(T, E_n)$ for almost all $\tau \in T$ and [12, Lemma 16, p. 196] $\tau \to \tilde{f}(\cdot, \tau, y(\tau), \omega^\square)$ is a measurable function from T to $L^p(T, E_n)$. Furthermore,

$$\tau \to 1 + |y(\tau)|^\beta \in L^{p/\beta}(T) \qquad \text{and} \qquad \tau \to |\psi_0(\cdot, \tau)|_p \in L^{p/(p-\beta)}(T);$$

hence, by Hölder's inequality,

$$\int |\tilde{f}(\cdot, \tau, y(\tau), \omega^\square)|_p \, d\tau \leqslant \int |\psi_0(\cdot, \tau)|_p \, (1 + |y(\tau)|^\beta) \, d\tau < \infty.$$

Thus $\tau \to \tilde{f}(\cdot, \tau, y(\tau), \omega^\square)$ is an integrable function from T to $L^p(T, E_n)$ for all $y \in L^p(T, E_n)$ and $\omega^\square \in \Omega$, and $\tilde{F}$ exists on $L^p(T, E_n) \times \Omega$.

Now consider the continuity of $\tilde{F}$ and the existence and continuity of $\tilde{F}_y$. We have, for fixed $y \in \mathscr{Y}$ and $\omega^\square \in \Omega$, and for all $\Delta y \in \mathscr{Y}$,

$$\begin{aligned} &\tilde{F}(y + \Delta y, \omega^\square)(t) - \tilde{F}(y, \omega^\square)(t) \\ &\quad = \int \{\tilde{f}(t, \tau, y(\tau) + \Delta y(\tau), \omega^\square) - \tilde{f}(t, \tau, y(\tau), \omega^\square)\} \, d\tau \qquad (7.1.1) \\ &\quad = \int \tilde{f}_v(t, \tau, y(\tau) + \theta(t, \tau) \, \Delta y(\tau), \omega^\square) \, \Delta y(\tau) \, d\tau \qquad \text{a.e. in } T, \end{aligned}$$

where $0 \leqslant \theta(t, \tau) \leqslant 1$, and we may assume (using essentially the argument in [13, Lemma 18.1, p. 177]) that $(t, \tau) \to \theta(t, \tau)$ is measurable. Furthermore,

$$|\tilde{f}_v(t, \tau, y(\tau) + \theta(t, \tau) \, \Delta y(\tau), \omega^\square)| \leqslant (1 + (|y(\tau)| + |\Delta y(\tau)|)^\alpha) \, \psi_1(t, \tau);$$

hence

$$\begin{aligned} &\int |\tilde{f}_v(t, \tau, y(\tau) + \theta(t, \tau) \, \Delta y(\tau), \omega^\square)|^p \, dt \\ &\qquad \leqslant (1 + (|y(\tau)| + |\Delta y(\tau)|)^\alpha)^p \, |\psi_1(\cdot, \tau)|_p^p . \end{aligned}$$

It follows that $t \to \tilde{f}_v(t, \tau, y(\tau) + \theta(t, \tau) \, \Delta y(\tau), \omega^\square)$ belongs to $L^p(T, E_{n^2})$ for almost all $\tau \in T$ and

$$\begin{aligned} &|\tilde{F}(y + \Delta y, \omega^\square) - \tilde{F}(y, \omega^\square)|_p \\ &\qquad \leqslant \int (1 + (|y(\tau)| + |\Delta y(\tau)|)^\alpha) \, |\psi_1(\cdot, \tau)|_p \, |\Delta y(\tau)| \, d\tau. \end{aligned}$$

We can easily verify that, for a fixed y in $\mathscr{Y}$, the coefficient of $|\Delta y(\tau)|$ in the integrand on the right has an $L^{p/(p-1)}$ norm bounded by some constant c_1 for all Δy in the unit ball of $L^p(T, E_n)$.

We conclude that

$$|\tilde{F}(y + \Delta y, \omega^\square) - \tilde{F}(y, \omega^\square)|_p \leqslant c_1 | \Delta y |_p$$

for all $\Delta y \in L^p(T, E_n)$ and $\omega^\square \in \Omega$. Thus $y \to \tilde{F}(y, \omega^\square)$ is continuous at every y, uniformly in $\omega^\square \in \Omega$.

Our previous argument shows that the function

$$\Delta y \to \mathscr{D}_1(\Delta y; y) = \int \tilde{f}_v(\cdot, \tau, y(\tau), \omega^\square)\, \Delta y(\tau)\, d\tau$$

is a bounded linear operator on $L^p(T, E_n)$ for every $(y, \omega^\square)$. Relation (7.1.1) now yields

$$\begin{aligned}|(\tilde{F}(y + \Delta y, \omega^\square) &- \tilde{F}(y, \omega^\square) - \mathscr{D}_1(\Delta y; y))(t)| \\ \leqslant \int \operatorname*{Sup}_{R\times\Omega} &| g_v(t, \tau, y(\tau) + \theta(t, \tau)\, \Delta y(\tau), r, \tilde{b}(\omega^\square)) \\ &- g_v(t, \tau, y(\tau), r, \tilde{b}(\omega^\square))|\, | \Delta y(\tau)|\, d\tau. \qquad (7.1.2)\end{aligned}$$

As Δy converges to 0 in $L^p(T, E_n)$, hence also in measure, the coefficient of $| \Delta y(\tau)|$ in the integrand on the right converges to 0 in measure, as a function of τ, for almost all $t \in T$. This coefficient is also bounded by $\alpha(t, \tau) = \psi_1(t, \tau)(2 + | y(\tau)|^\alpha + (| y(\tau)| + | \Delta y(\tau)|)^\alpha)$, and we verify that $t \to | \alpha(t, \cdot)|_{p/(p-1)}$ belongs to $L^p(T)$. It follows, applying Hölder's inequality to the right side of (7.1.2) and then taking the L^p-norm with respect to t, that

$$\lim_{|\Delta y|_p \to 0} | \tilde{F}(y + \Delta y, \omega^\square) - \tilde{F}(y, \omega^\square) - \mathscr{D}_1(\Delta y; y)|_p / | \Delta y |_p = 0$$

for every $y \in L^p(T, E_n)$, uniformly in $\omega^\square \in \Omega$; hence

$$\mathscr{D}_1(\Delta y; y) = \tilde{F}_y(y, \omega^\square)\, \Delta y \qquad (\Delta y \in L^p(T, E^n)),$$

and $\tilde{F}_y(y, \omega^\square)$ is the operator $\Delta y \to \int_T \tilde{f}_v(\cdot, \tau, y(\tau), \omega^\square)\, \Delta y(\tau)\, d\tau$. Thus $\tilde{F}_y(y, \omega^\square)$ and $\tilde{F}_y(\bar{y}, 0^\square)$ have the form indicated in the statement of the Lemma.

The argument we have used to prove the existence of $\tilde{F}_y$ via inequality (7.1.2) and Assumption (4.1.4) can be used to show that $| \tilde{F}_y(y_1, \omega^\square) - \tilde{F}_y(y_2, \omega^\square)| \to 0$ as $y_2 \to y_1$ in $\mathscr{Y}$, uniformly in $\omega^\square$. Thus $y \to \tilde{F}(y, \omega^\square)$ and $y \to \tilde{F}_y(y, \omega^\square)$ are continuous at each y, uniformly in $\omega^\square \in \Omega$. Similar arguments show that $\omega^\square \to \tilde{F}(y, \omega^\square)$ and $\omega^\square \to \tilde{F}_y(y, \omega^\square)$ are continuous for each $y \in \mathscr{Y}$, whence we conclude that $(y, \omega^\square) \to \tilde{F}(y, \omega^\square)$ and $(y, \omega^\square) \to \tilde{F}_y(y, \omega^\square)$ exist and are continuous on $\mathscr{Y} \times \Omega$. Finally, the existence of $\tilde{F}_{\omega^\square}(y, 0^\square)$ follows from that of $\tilde{f}_{\omega^\square}(t, \tau, y(\tau), 0^\square)$ and the bounds in (4.1.4). Thus $(y, \omega^\square) \to \tilde{F}(y, \omega^\square)$ has a (total) derivative at $(\bar{y}, 0^\square)$.

The same conclusions can be reached by similar arguments when $\mathscr{Y} = C(T, E_n)$. Q.E.D.

LEMMA 7.2. *The mapping* $I - F_y(\bar{y}, \bar{\sigma}, \bar{b})$ *is a linear homeomorphism of* $\mathscr{Y}$ *onto* $\mathscr{Y}$, *and statement* (4.2.1) *is valid.*

Proof. We have shown in Lemma 7.1 that

$$(F_y(\bar{y}, \bar{\sigma}, \bar{b})\, \Delta y)(t) = \int k(t, \tau)\, \Delta y(\tau)\, d\tau \qquad (t \in T, \Delta y \in \mathscr{Y}).$$

By Assumption 4.1, k is measurable on $T \times T$ and $|\, k(t, \tau)|$ is bounded by $\tilde{\psi}(t, \tau) = (1 + |\,\bar{y}(\tau)|^\alpha)\, \psi_1(t, \tau)$ for $\mathscr{Y} = L^p(T, E_n)$. We verify then, as in Lemma 7.1, that $\int |\, \tilde{\psi}(\cdot, \tau)|_p^{p/(p-1)}\, d\tau < \infty$. It follows [*12*, p. 518] that $F_y(\bar{y}, \bar{\sigma}, \bar{b})$ is a compact operator on $L^p(T, E_n)$. Similarly, if $\mathscr{Y} = C(T, E_n)$, the family of functions

$$t \to \int k(t, \tau)\, \Delta y(\tau)\, d\tau$$

corresponding to all Δy such that $\mathrm{Max}_{t \in T} |\, \Delta y(t)| \leqslant 1$ is uniformly bounded and has the common modulus of continuity Φ. Thus, in both cases, $F_y(\bar{y}, \bar{\sigma}, \bar{b})$ is a compact operator. It follows, therefore, from (4.1.5) that $I - F_y(\bar{y}, \bar{\sigma}, \bar{b})$ is a linear homeomorphism of $\mathscr{Y}$ onto $\mathscr{Y}$ [*12*, Theorem 5, p. 579].

Let $K = F_y(\bar{y}, \bar{\sigma}, \bar{b})$ and $K^* = (I - K)^{-1} - I$. For $\mathscr{Y} = L^p(T, E_n)$, the arguments of [*14*, pp. 157-160] (applying to the case $n = 1, p = 2$) can be suitably generalized to prove that K^* is an integral operator such that $(K^* \Delta y)(t) = \int k^*(t, \tau)\, \Delta y(\tau)\, d\tau (t \in T, \Delta y \in \mathscr{Y})$, where k^* is as described in (4.2.1). (These arguments, in their generalized form, are based on approximating the function $\tau \to k(\cdot, \tau)$ in $L^{p/(p-1)}(T, L^p(T, E_n))$ by finite sums of the form $\sum \alpha_j(\tau)\, \beta_j(\cdot)$). Finally, since $k_j^{\,i}(i = 1,..., m)$ are independent of t and $K^* = K + KK^*$, the $k_j^{*i}(i = 1,..., m)$ are also independent of t.

For $\mathscr{Y} = C(T, E_n)$, we observe that since $K^* = K + KK^*$ and K is compact, so is K^*. There exist, therefore [*15*, Proposition 9.5.17, p. 665], a measurable $k^\# = (k_j^{\# i})(i, j = 1,..., n)$ on $T \times T$ and a nonnegative regular Borel measure μ on T such that

$$(K^* \Delta y)(t) = \int k^\#(t, \tau)\, \Delta y(\tau)\, \mu(d\tau) \qquad (t \in T, \Delta y \in \mathscr{Y})$$

and $\operatorname{Sup}_{t \in T} \int_T | k^*(t, \tau)| \mu(d\tau) < \infty$. Our conclusions about k^* will follow directly from the Radon–Nikodym theorem once we prove that, for all $t \in T$, the measure $A \to \int_A k^*(t, \tau) \mu(d\tau)$ is absolutely continuous with respect to our original measure $A \to \int_A d\tau$. This we can do by observing that if $K \Delta y_i \to_{i \to \infty} 0$ in $\mathscr{Y}$, so does $K^* \Delta y_i = (I + K^*) K \Delta y_i$; and then considering any sequence $\{A_i\}$ of Borel sets in T such that $| A_i | \to_{i \to \infty} 0$ and "approximating" their characteristic functions with continuous functions a_i such that $0 \leqslant a_i(t) \leqslant 1$, $a_i(t) = 1$ on C_i, $a_i(t) = 0$ on $T - G_i$, where $C_i \subset A_i \subset G_i$, C_i are closed, G_i are open, and

$$\mu(G_i - C_i) + | G_i - C_i | \to 0 \quad \text{as} \quad i \to \infty.$$

Q.E.D.

7.3 *Completion of Proof of Theorem 4.2.* Lemmas 7.1 and 7.2 show that Theorem 2.1 is applicable to the control problem as defined in Section 4, and that statement (4.2.1) is valid. We have, for $q = (\sigma, b)$, $\sigma_{11} = \sigma$, $b_{11} = b$,

$$DF(\bar{y}, \bar{q}; q - \bar{q}) = DF(\bar{y}, \bar{\sigma}, \bar{b}; (\sigma, b) - (\bar{\sigma}, \bar{b})) = \bar{F}_{\omega^{11}}(\bar{y}, 0)$$

and

$$c^i(y, q) = F^i(y, q) \qquad (i = 1, \ldots, m).$$

Let K^* be the linear operator on $\mathscr{Y}$ defined by k^* and let $\hat{\lambda} = (\lambda, 0, \ldots, 0) \in E_n$. Then $K^* = (I - F_y(\bar{y}, \bar{q}))^{-1} - I$ and, applying Lemma 7.1, relation (2.3.2) can be rewritten as

$$\begin{aligned}
&\hat{\lambda} \cdot \{F_y(\bar{y}, \bar{q})(I - F_y(\bar{y}, \bar{q}))^{-1} DF(\bar{y}, \bar{q}; q - \bar{q}) + DF(\bar{y}, \bar{q}; q - \bar{q})\} \\
&\quad = \hat{\lambda} \cdot (I + K^*) DF(\bar{y}, \bar{q}; q) \\
&\quad = \sum_{j=1}^{m} \lambda^j \int_T \{f^j(\theta, \bar{y}(\theta), \sigma(\theta) - \bar{\sigma}(\theta), \bar{b}) + Df^j(\theta, \bar{y}(\theta), \bar{\sigma}(\theta), \bar{b}; b - \bar{b})\} \, d\theta \\
&\qquad + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda^i \int_{T \times T} k_j^{*i}(\tau) \{f^j(\tau, \theta, \bar{y}(\theta), \sigma(\theta) - \bar{\sigma}(\theta), \bar{b}) \\
&\qquad + Df^j(\tau, \theta, \bar{y}(\theta), \bar{\sigma}(\theta), \bar{b}; b - \bar{b})\} \, d\tau d\theta \\
&\quad = \int_T d\theta \int_T \zeta(\tau) \cdot \{f(\tau, \theta, y(\theta), \sigma(\theta) - \bar{\sigma}(\theta), \bar{b}) \\
&\qquad + Df(\tau, \theta, \bar{y}(\theta), \bar{\sigma}(\theta), \bar{b}; b - \bar{b})\} \, d\tau \\
&\quad = \int_T H_1(\sigma(\theta), \theta) \, d\theta - \int_T H_1(\bar{\sigma}(\theta), \theta) \, d\theta + H_2(b) \geqslant 0
\end{aligned} \tag{7.3.1}$$

for all $(\sigma, b) \in \mathscr{S} \times B$. In particular, for $\sigma = \bar{\sigma}$, $H_2(b) \geqslant 0 = H_2(\bar{b})$ for all $b \in B$.

It remains now to prove relation (1). Let $R_\infty = \{r_1, r_2, ...\}$ be a dense subset of R, s_{r_i} a measure concentrated at r_i with mass 1, $i \in \{1, 2, ...\}$, E an arbitrary measurable subset of T, $b = \bar{b}$, and $\sigma(t) = s_{r_i}$ for $t \in E$, $\sigma(t) = \bar{\sigma}(t)$ for $t \in T - E$. Then relation (7.3.1) yields

$$\int_E \{H_1(r_i, \theta) - H_1(\bar{\sigma}(\theta), \theta)\}\, d\theta \geqslant 0,$$

where $H_1(r, \theta) = H_1(s_r, \theta)$ and s_r is a measure concentrated at r with mass 1. It follows that for each i there exists a subset T_i of T, of measure $| T |$, such that

$$H_1(r_i, \theta) \geqslant H_1(\bar{\sigma}(\theta), \theta) \qquad \text{for all} \quad \theta \in T_i .$$

Then, for $T' = \bigcap_{i=1}^{\infty} T_i$,

$$H_1(r, \theta) = \int_T \zeta(\tau) \cdot g(\tau, \theta, \bar{y}(\theta), r, \bar{b})\, d\tau \geqslant H_1(\bar{\sigma}(\theta), \theta) \tag{7.3.2}$$

for all $\theta \in T'$ and $r \in R_\infty$. We verify, using properties of k^* described in (4.2.1) and the bounds on g described in Assumption (4.1.4), that $\tau \to \zeta(\tau) \cdot g(\tau, \theta, \bar{y}(\theta), r, \bar{b})$ is bounded for all r and almost all θ by an integrable function of τ. Since, furthermore, it is also continuous in r, we conclude that relation (7.3.2) is valid for almost all θ and all $r \in R$ and, integrating both sides with respect to any $s \in S$, that relation (1) is valid. Q.E.D.

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